ON THE HOCHSCHILD (CO)HOMOLOGY OF QUANTUM EXTERIOR ALGEBRAS

PETTER ANDREAS BERGH

Abstract. We compute the Hochschild cohomology and homology of the algebra $\Lambda = k\langle x,y\rangle/(x^2, xy+qyx, y^2)$ with coefficients in ${}_1\Lambda_\psi$ for every degree preserving k-algebra automorphism $\psi \colon \Lambda \to \Lambda$. As a result we obtain several interesting examples of the homological behavior of Λ as a bimodule.

Introduction

Throughout this paper, let k be a field and $q \in k$ a nonzero element which is *not* a root of unity. Denote by Λ the k-algebra

$$\Lambda = k\langle x,y\rangle/(x^2, xy+qyx, y^2),$$

and by Λ^{e} its enveloping algebra $\Lambda^{\mathrm{op}} \otimes_k \Lambda$. All modules considered are assumed to be right modules.

During the last years, this 4-dimensional graded Koszul algebra, whose module category was classified in [7], has provided several examples (or rather counterexamples) giving negative answers to homological conjectures and questions. Among these are the conjecture of Auslander on local Ext-limitations (see [1, page 815], [6] and [8]) and the question of Happel on the relation between the global dimension and the vanishing of the Hochschild cohomology (see [5] and [2]).

We shall study the Hochschild cohomology and homology of Λ. More precisely, for every degree preserving k-algebra automorphism $\psi \colon \Lambda \to \Lambda$ we compute $\mathrm{HH}^*(\Lambda, {}_1\Lambda_\psi) = \mathrm{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda, {}_1\Lambda_\psi)$ and $\mathrm{HH}_*(\Lambda, {}_1\Lambda_\psi) = \mathrm{Tor}^{\Lambda^{\mathrm{e}}}_*(\Lambda, {}_1\Lambda_\psi)$, that is, the Hochschild cohomology and homology of Λ with coefficients in the twisted bimodule ${}_1\Lambda_\psi$ (the action of Λ^{e} on ${}_1\Lambda_\psi$ is defined as $\lambda(\lambda_1 \otimes \lambda_2) = \lambda_1\lambda\psi(\lambda_2)$). As a result we obtain several interesting examples, both in cohomology and homology, of the homological behavior of Λ as a bimodule.

1. The Hochschild Homology

Denote by D the usual k-dual $\mathrm{Hom}_k(-,k)$, and consider the map $\phi \colon {}_\Lambda\Lambda \to D(\Lambda_\Lambda)$ of left Λ-modules defined by

$$\phi(1)(\alpha + \beta x + \gamma y + \delta yx) \stackrel{\text{def}}{=} \delta.$$

It is easy to show that this is an injective map and hence also an isomorphism since $\dim_k \Lambda = \dim_k D(\Lambda)$, and therefore Λ is a Frobenius algebra by definition. Now take any element $\lambda \in \Lambda$, and consider the element $\phi(1) \cdot \lambda \in D(\Lambda)$ (we consider $D(\Lambda)$ as a Λ-Λ-bimodule). As ϕ is surjective, there is an element $\lambda' \in \Lambda$ such that $\lambda' \cdot \phi(1) = \phi(\lambda') = \phi(1) \cdot \lambda$, and the map $\lambda \mapsto \lambda'$ defines a k-algebra automorphism $\nu^{-1} \colon \Lambda \to \Lambda$ whose inverse ν is called the *Nakayama automorphism* of Λ (with respect to the map ϕ). Straightforward calculations show that $x \cdot \phi(1) =$

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$\phi(1)\cdot(-q^{-1}x)$ and $y\cdot\phi(1)=\phi(1)\cdot(-qy)$, hence since x and y generate Λ over k we see that ν is the degree preserving map defined by

$$x\mapsto -q^{-1}x, \quad y\mapsto -qy.$$

The map ϕ induces a bimodule isomorphism ${}_1\Lambda_{\nu^{-1}}\simeq D(\Lambda)$, which in turn gives an isomorphism ${}_{(\nu\psi)^{-1}}\Lambda_{\nu^{-1}}\simeq_{(\nu\psi)^{-1}} D(\Lambda)_1$ for any automorphism ψ of Λ. Furthermore, since ${}_{(\nu\psi)^{-1}}\Lambda_{\nu^{-1}}$ is isomorphic to ${}_1\Lambda_\psi$ and ${}_{(\nu\psi)^{-1}}D(\Lambda)_1=D({}_1\Lambda_{(\nu\psi)^{-1}})$, we get an isomorphism ${}_1\Lambda_\psi\simeq D({}_1\Lambda_{(\nu\psi)^{-1}})$ of bimodules. Now from [3, Proposition VI.5.1] we get

$$\begin{aligned}\mathrm{HH}^n(\Lambda,{}_1\Lambda_\psi) &= \mathrm{Ext}^n_{\Lambda^{\mathrm{e}}}(\Lambda,{}_1\Lambda_\psi)\\ &\simeq \mathrm{Ext}^n_{\Lambda^{\mathrm{e}}}\left(\Lambda, D({}_1\Lambda_{(\nu\psi)^{-1}})\right)\\ &\simeq D\left(\mathrm{Tor}_n^{\Lambda^{\mathrm{e}}}(\Lambda,{}_1\Lambda_{(\nu\psi)^{-1}})\right)\\ &= D\left(\mathrm{HH}_n(\Lambda,{}_1\Lambda_{(\nu\psi)^{-1}})\right),\end{aligned}$$

thus when computing the (dimension of the) Hochschild cohomology group $\mathrm{HH}^n(\Lambda,{}_1\Lambda_\psi)$ we are also computing the Hochschild homology group $\mathrm{HH}_n(\Lambda,{}_1\Lambda_{(\nu\psi)^{-1}})$. Moreover, as ψ ranges over all degree preserving k-algebra automorphisms of Λ, so does $(\nu\psi)^{-1}$.

2. The Cohomology Complex

We start by recalling the construction of the minimal bimodule projective resolution of Λ from [2]. Define the elements

$$f_0^0=1, \quad f_0^1=x, \quad f_1^1=y,$$
$$f_{-1}^n=0=f_{n+1}^n \ \text{ for each } n\geq 0,$$

and for each $n\geq 2$ define elements $\{f_i^n\}_{i=0}^n\subseteq \underbrace{\Lambda\otimes_k\cdots\otimes_k\Lambda}_{n\text{ copies}}$ inductively by

$$f_i^n=f_{i-1}^{n-1}\otimes y+q^i f_i^{n-1}\otimes x.$$

Denote by P^n the Λ^{e}-projective module $\bigoplus_{i=0}^n\Lambda\otimes_k f_i^n\otimes_k\Lambda$, and by $\tilde{f}_i^n$ the element $1\otimes f_i^n\otimes 1\in P^n$ (and $\tilde{f}_0^0=1\otimes 1$). The set $\{\tilde{f}_i^n\}_{i=0}^n$ generates P^n as a Λ^{e}-module. Now define a map $\delta_n\colon P^n\to P^{n-1}$ by

$$\tilde{f}_i^n\mapsto\left[x\tilde{f}_i^{n-1}+(-1)^n q^i\tilde{f}_i^{n-1}x\right]+\left[q^{n-i}y\tilde{f}_{i-1}^{n-1}+(-1)^n\tilde{f}_{i-1}^{n-1}y\right].$$

It is shown in [2] that

$$(\mathbb{P},\delta)\colon\ \cdots\to P^{n+1}\xrightarrow{\delta_{n+1}}P^n\xrightarrow{\delta_n}P^{n-1}\to\cdots$$

is a minimal Λ^{e}-projective resolution of Λ. Denote the direct sum of n copies of ${}_1\Lambda_\psi$ by ${}_1\Lambda_\psi^n$, and consider its standard k-basis $\{e_i^{n-1},xe_i^{n-1},ye_i^{n-1},yxe_i^{n-1}\}_{i=0}^{n-1}$. Define a map $d_n\colon{}_1\Lambda_\psi^n\to{}_1\Lambda_\psi^{n+1}$ by

$$\lambda e_i^{n-1}\mapsto\left[x\lambda+(-1)^n q^i\lambda\psi(x)\right]e_i^n+\left[q^{n-i-1}y\lambda+(-1)^n\lambda\psi(y)\right]e_{i+1}^n.$$

Applying $\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(-,{}_1\Lambda_\psi)$ to the resolution $(\mathbb{P},\delta)$, keeping in mind that $\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P^n,{}_1\Lambda_\psi)$ and ${}_1\Lambda_\psi^{n+1}$ are isomorphic as k-vector spaces, we get the commutative diagram

$$\begin{tikzcd}\cdots \arrow[r] & \mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P^{n-1},{}_1\Lambda_\psi) \arrow[r, "\delta_n^*"] \arrow[d, "\wr"] & \mathrm{Hom}_{\Lambda^{\mathrm{e}}}(P^n,{}_1\Lambda_\psi) \arrow[r, "\delta_{n+1}^*"] \arrow[d, "\wr"] & \cdots\\ \cdots \arrow[r] & {}_1\Lambda_\psi^n \arrow[r, "d_n"] & {}_1\Lambda_\psi^{n+1} \arrow[r, "d_{n+1}"] & \cdots\end{tikzcd}$$

of k-vector spaces.

In order to compute $\mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda, {}_1\Lambda_\psi)$ for $n > 0$ we compute the cohomology $\operatorname{Ker} d_{n+1}/\operatorname{Im} d_n$ of the bottom complex in the above commutative diagram. We do this by finding $\dim_k \operatorname{Im} d_n$; once we know $\dim_k \operatorname{Im} d_n$, we obtain $\dim_k \operatorname{Ker} d_n$ (and therefore also $\dim_k \operatorname{Ker} d_{n+1}$) from the equation

$$\dim_k \operatorname{Ker} d_n + \dim_k \operatorname{Im} d_n = \dim_k {}_1\Lambda^n_\psi = 4n.$$

We then have

$$\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = \dim_k \operatorname{Ker} d_{n+1} - \dim_k \operatorname{Im} d_n.$$

Now let ψ be a degree preserving k-algebra automorphism of Λ. Then there are elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in k$ such that $\psi(x) = \alpha_1 x + \alpha_2 y$ and $\psi(y) = \beta_1 x + \beta_2 y$. Since $\psi(x^2) = \psi(y^2) = \psi(xy + qyx) = 0$, we have the relations $\alpha_1\alpha_2 = \beta_1\beta_2 = \alpha_2\beta_1 = 0$. If $\alpha_2 \neq 0$, then $\alpha_1 = \beta_1 = 0$, implying $x \notin \operatorname{Im}\psi$. Similarly, if $\beta_1 \neq 0$, then $\alpha_2 = \beta_2 = 0$, implying $y \notin \operatorname{Im}\psi$. Therefore $\alpha_2 = \beta_1 = 0$, and this forces α_1 and β_2 to be nonzero. Thus the degree preserving k-algebra automorphisms of Λ are precisely those defined by

$$x \mapsto \alpha x, \quad y \mapsto \beta y$$

for two arbitrary nonzero elements $\alpha, \beta \in k$. For such an automorphism, the result of applying d_n to the basis vectors $\{e_i^{n-1}, xe_i^{n-1}, ye_i^{n-1}, yxe_i^{n-1}\}_{i=0}^{n-1}$ of ${}_1\Lambda^n_\psi$ is

$$\begin{array}{rcl} e_i^{n-1} & \mapsto & \left[1 + (-1)^n q^i \alpha\right] xe_i^n + \left[q^{n-i-1} + (-1)^n \beta\right] ye_{i+1}^n \\ xe_i^{n-1} & \mapsto & \left[q^{n-i-1} + (-1)^{n+1} q\beta\right] yxe_{i+1}^n \\ ye_i^{n-1} & \mapsto & \left[-q + (-1)^n q^i \alpha\right] yxe_i^n \\ yxe_i^{n-1} & \mapsto & 0 \end{array}$$

for $0 \leq i \leq n-1$. Note that the inequality $\dim_k \operatorname{Im} d_n \leq 2n+1$ always holds.

3. The Hochschild Cohomology

We start by computing $\mathrm{HH}^0(\Lambda, {}_1\Lambda_\psi)$. Rather than computing this vector space directly using the identifications

$$\begin{array}{rcl} \mathrm{HH}^0(\Lambda, {}_1\Lambda_\psi) & = & \{z \in {}_1\Lambda_\psi \mid \lambda \cdot z = z \cdot \lambda \text{ for all } \lambda \in \Lambda\} \\ & = & \{z \in {}_1\Lambda_\psi \mid \lambda z = z\psi(\lambda) \text{ for all } \lambda \in \Lambda\} \\ & = & \{z \in {}_1\Lambda_\psi \mid xz = z\psi(x), yz = z\psi(y), yxz = z\psi(yx)\} \\ & = & \{z \in {}_1\Lambda_\psi \mid xz = \alpha zx, yz = \beta zy, yxz = \alpha\beta zyx\}, \end{array}$$

we use our cohomology complex and the isomorphism $\mathrm{HH}^0(\Lambda, {}_1\Lambda_\psi) \simeq \operatorname{Ker} d_1$. From the above we see that the map d_1 is defined by

$$\begin{array}{rcl} e_0^0 & \mapsto & [1-\alpha]xe_0^1 + [1-\beta]ye_1^1 \\ xe_0^0 & \mapsto & [1+q\beta]yxe_1^1 \\ ye_0^0 & \mapsto & -[q+\alpha]yxe_0^1 \\ yxe_0^0 & \mapsto & 0, \end{array}$$

and so calculation gives

$$\dim_k \mathrm{HH}^0(\Lambda, {}_1\Lambda_\psi) = \begin{cases} 3 & \text{when } \alpha = -q, \beta = -q^{-1} \\ 2 & \text{when } \alpha = 1, \beta = 1 \\ 2 & \text{when } \alpha = -q, \beta \neq -q^{-1} \\ 2 & \text{when } \alpha \neq -q, \beta = -q^{-1} \\ 1 & \text{otherwise} \end{cases}$$

when the characteristic of k is not 2. In the characteristic 2 case we replace $-q, -q^{-1}$ and 1 in the above formula by $\pm q, \pm q^{-1}$ and ± 1, respectively.

Now we turn to the cohomology groups $\mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi)$ for $n > 0$. To compute the dimension of $\operatorname{Im} d_n$, we distinguish between four possible cases depending on whether or not α and β belong to the set

$$\Sigma = \{\pm q^i\}_{i \in \mathbb{Z}}.$$

3.1. **The case** $\alpha, \beta \notin \Sigma$**:**
This is the easiest case; $d_n(e_i^{n-1}), d_n(xe_i^{n-1})$ and $d_n(ye_i^{n-1})$ are all nonzero for $0 \leq i \leq n-1$, hence $\dim_k \operatorname{Im} d_n = 2n+1$ for all n. Then $\dim_k \operatorname{Ker} d_n = 2n-1$, implying $\dim_k \operatorname{Ker} d_{n+1} = 2n+1$ and therefore that $\mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 0$ for $n > 0$.

Remark. We can relate the vanishing of cohomology to the conjecture of Tachikawa stating that over a selfinjective ring the only finitely generated modules having no self extensions are the projective ones. Namely, let M be a finitely generated Λ-module such that Λ has no bimodule extensions by $\operatorname{Hom}_k(M, M)$. Then from [3, Corollary IX.4.4] we get

$$\operatorname{Ext}_\Lambda^n(M, M) \simeq \mathrm{HH}^n(\Lambda, \operatorname{Hom}_k(M, M)) = 0$$

for $n > 0$. Since Tachikawa's conjecture holds for Λ (see [7, Proposition 4.2]), the module M must be projective and therefore (Λ is local) isomorphic to Λ^t for some $t \in \mathbb{N}$. This gives

$$\operatorname{Hom}_k(M, M) \simeq (\Lambda \otimes_k D(\Lambda))^{t^2} \simeq (\Lambda^{\mathrm{e}})^{t^2}.$$

In particular, if $\mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 0$ for $n > 0$ then there cannot exist a Λ-module M such that $\operatorname{Hom}_k(M, M)$ is isomorphic to ${}_1\Lambda_\psi$, since this would imply the contradiction ${}_1\Lambda_\psi \simeq (\Lambda^{\mathrm{e}})^{t^2}$.

3.2. **The case** $\alpha \in \Sigma, \beta \notin \Sigma$**:**
Since $\left[q^{n-i-1} + (-1)^n \beta\right]$ and $\left[q^{n-i-1} + (-1)^{n+1} q\beta\right]$ are nonzero, we have $d_n(e_i^{n-1}) \neq 0$ and $d_n(xe_i^{n-1}) \neq 0$ for $0 \leq i \leq n-1$. Therefore $\dim_k \operatorname{Im} d_n \geq 2n$, and the problem is now whether or not the basis vector yxe_0^n belongs to $\operatorname{Im} d_n$. This is the case if and only if $d_n(ye_0^{n-1}) \neq 0$, that is, if and only if

$$-q + (-1)^n \alpha \neq 0 \tag{C1}$$

holds. We now break down this case into three cases.

(i) The case $\alpha \neq \pm q$*:*
Since (C1) holds we have $\dim_k \operatorname{Im} d_n = 2n+1$, and so $\mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 0$ for $n > 0$.

(ii) The case $\alpha = q$*:*
When the characteristic of k is not 2, the condition (C1) holds if and only if n is odd. Therefore

$$\dim_k \operatorname{Im} d_n = \begin{cases} 2n & \text{for } n \text{ even} \\ 2n+1 & \text{for } n \text{ odd,} \end{cases}$$

and so

$$\dim_k \operatorname{Ker} d_{n+1} = \begin{cases} 2n+1 & \text{for } n \text{ even} \\ 2n+2 & \text{for } n \text{ odd.} \end{cases}$$

This gives $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 1$ for $n > 0$.

When k is of characteristic 2 we see that (C1) never holds, hence $\dim_k \operatorname{Im} d_n = 2n$. Then $\dim_k \operatorname{Ker} d_{n+1} = 2n+2$, giving $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$.

(iii) The case $\alpha = -q$:

As in the previous case, we get $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 1$ for $n > 0$ when k is not of characteristic 2, and $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$ in the characteristic 2 case.

Remark. From this case we obtain an example showing that symmetry in the vanishing of Ext over Λ^{e} does not hold. Namely, define ψ by

$$x \mapsto q^{-1}x, \quad y \mapsto \beta y$$

for some element β not contained in Σ. For $n > 0$, case (i) above gives $\mathrm{Ext}^n_{\Lambda^{\mathrm{e}}}(\Lambda, {}_1\Lambda_\psi) = 0$, whereas from case (ii) we see that $\dim_k \mathrm{Ext}^n_{\Lambda^{\mathrm{e}}}(\Lambda, {}_1\Lambda_{\psi^{-1}})$ is either 1 or 2, depending on the characteristic of k. Now it is easy to see that $\mathrm{Ext}^n_{\Lambda^{\mathrm{e}}}(\Lambda, {}_1\Lambda_{\psi^{-1}})$ is isomorphic to $\mathrm{Ext}^n_{\Lambda^{\mathrm{e}}}({}_1\Lambda_\psi, \Lambda)$.

3.3. **The case $\alpha \notin \Sigma, \beta \in \Sigma$:**

The algebra Λ is isomorphic to the algebra $k\langle u, v\rangle/(u^2, uv + q^{-1}vu, v^2)$ via the map

$$x \mapsto v, \quad y \mapsto u,$$

hence this case is symmetric to the case $\alpha \in \Sigma, \beta \notin \Sigma$ treated above. Namely, when $\beta \neq \pm q^{-1}$ the result is as in (i), whereas when $\beta = \pm q^{-1}$ the result is as in (ii) and (iii).

3.4. **The case $\alpha \in \Sigma, \beta \in \Sigma$:**

The basis vectors yxe_0^n and yxe_n^n can only be the image of ye_0^{n-1} and xe_{n-1}^{n-1}, respectively, whereas for $1 \leq i \leq n-1$ the basis vector yxe_i^n can be the image of both ye_i^{n-1} and xe_{i-1}^{n-1}. Therefore we break this case down into four cases, each depending on whether or not $\alpha = \pm q$ and $\beta = \pm q^{-1}$.

(i) The case $\alpha = \pm q, \beta = \pm q^{-1}$:

We have

$$d_n(e_i^{n-1}) = \left[1 \pm (-1)^n q^{i+1}\right] xe_i^n + \left[q^{n-i-1} \pm (-1)^n q^{-1}\right] ye_{i+1}^n,$$

and since $i + 1 \geq 1$ the term $\left[1 \pm (-1)^n q^{i+1}\right]$ must be nonzero. Therefore $d_n(e_i^{n-1}) \neq 0$. Applying d_n to xe_i^{n-1} and ye_i^{n-1} gives $\left[q^{n-i-1} \pm (-1)^{n+1}\right] yxe_{i+1}^n$ and $\left[-q \pm (-1)^n q^{i+1}\right] yxe_i^n$, respectively, hence when the characteristic of k is not 2 we get

$$d_n(xe_i^{n-1}) = \begin{cases} 0 & \text{for } i = n-1, \beta = q^{-1}, n \text{ even} \\ 0 & \text{for } i = n-1, \beta = -q^{-1}, n \text{ odd} \\ \neq 0 & \text{otherwise,} \end{cases}$$

$$d_n(ye_i^{n-1}) = \begin{cases} 0 & \text{for } i = 0, \alpha = q, n \text{ even} \\ 0 & \text{for } i = 0, \alpha = -q, n \text{ odd} \\ \neq 0 & \text{otherwise.} \end{cases}$$

There are four possible pairs (α, β) to consider. If $\alpha = q$ and $\beta = q^{-1}$, then the above gives

$$\dim_k \mathrm{Im}\, d_n = \begin{cases} 2n-1 & \text{for } n \text{ even} \\ 2n+1 & \text{for } n \text{ odd,} \end{cases}$$

and therefore

$$\dim_k \mathrm{Ker}\, d_{n+1} = \begin{cases} 2n+1 & \text{for } n \text{ even} \\ 2n+3 & \text{for } n \text{ odd.} \end{cases}$$

This implies $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$. Similar computation gives $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$ also for the other three possible pairs.

When k is of characteristic 2 then

$$d_n(xe_i^{n-1}) = \begin{cases} 0 & \text{for } i = n-1 \\ \neq 0 & \text{otherwise,} \end{cases}$$

$$d_n(ye_i^{n-1}) = \begin{cases} 0 & \text{for } i = 0 \\ \neq 0 & \text{otherwise,} \end{cases}$$

and so $\dim_k \operatorname{Im} d_n = 2n-1$ for all $n > 0$. Consequently $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 4$ for $n > 0$.

Remark. Note that, in this particular case, we have computed the (dimension of the) Hochschild homology $\mathrm{HH}_*(\Lambda) = \mathrm{Tor}_*^{\Lambda^e}(\Lambda, \Lambda)$ of Λ. Namely, it follows from Section 1 that for each $n > 0$ the k-vector spaces $\mathrm{HH}^n(\Lambda, {}_1\Lambda_{\nu^{-1}})$ and $D(\mathrm{HH}_n(\Lambda))$ are isomorphic, where ν is the Nakayama automorphism

$$x \mapsto -q^{-1}x, \quad y \mapsto -qy$$

of Λ. Then ν^{-1} is defined by

$$x \mapsto -qx, \quad y \mapsto -q^{-1}y,$$

hence ${}_1\Lambda_{\nu^{-1}}$ is precisely the sort of bimodule we have just considered in terms of Hochschild cohomology. Consequently $\dim_k \mathrm{HH}_n(\Lambda) = 2$ for $n > 0$ when the characteristic of k is not 2, whereas $\dim_k \mathrm{HH}_n(\Lambda) = 4$ for $n > 0$ in the characteristic 2 case.

(ii) The case $\alpha = \pm q, \beta \neq \pm q^{-1}$:
As in (i) the element $d_n(e_i^{n-1})$ is nonzero for $0 \leq i \leq n-1$. Moreover, since $\beta \neq \pm q^{-1}$ the basis element yxe_n^n always lies in $\operatorname{Im} d_n$, as does the basis elements yxe_i^n for $1 \leq i \leq n-1$. Therefore $\dim \operatorname{Im} d_n \geq 2n$, and the question is whether or not yxe_0^n belongs to $\operatorname{Im} d_n$, i.e. whether or not $d_n(ye_0^{n-1})$ is nonzero.
When the characteristic of k is not 2, then from (i) we see that

$$d_n(ye_0^{n-1}) = \begin{cases} 0 & \text{for } \alpha = q, n \text{ even} \\ 0 & \text{for } \alpha = -q, n \text{ odd} \\ \neq 0 & \text{otherwise,} \end{cases}$$

and computation gives $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 1$ for $n > 0$. However, when the characteristic of k is 2 then $d_n(ye_0^{n-1}) = 0$, giving $\dim_k \operatorname{Im} d_n = 2n$ and consequently $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$.

(iii) The case $\alpha \neq \pm q, \beta = \pm q^{-1}$:
Using the isomorphism $\Lambda \simeq k\langle u, v\rangle/(u^2, uv + q^{-1}vu, v^2)$ from the case $\alpha \notin \Sigma, \beta \in \Sigma$, we see that the present case is symmetric to the case (ii) above. Thus when the characteristic of k is not 2 then $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 1$ for $n > 0$, whereas $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$ in the characteristic 2 case.

(iv) The case $\alpha \neq \pm q, \beta \neq \pm q^{-1}$:
We now have $\alpha = \pm q^s$ and $\beta = \pm q^t$ where $s \in \mathbb{Z} \setminus \{1\}$ and $t \in \mathbb{Z} \setminus \{-1\}$, and therefore the basis elements yxe_0^n and yxe_n^n both lie in the image of d_n. To compute the dimension of $\operatorname{Im} d_n$, we must find out when $d_n(e_i^{n-1}) = 0$ for some $0 \leq i \leq n-1$ and when $yxe_i^n \notin \operatorname{Im} d_n$ for some $1 \leq i \leq n-1$. We have

$$(*) \qquad \begin{gathered} \left\{ d_n(e_i^{n-1}) = 0 \text{ for some } 0 \leq i \leq n-1 \right\} \\ \Updownarrow \\ \left\{ 1 + (-1)^n q^i \alpha = 0 \text{ and } q^{n-i-1} + (-1)^n \beta = 0 \right\} \end{gathered}$$

and

$$(**)\qquad \begin{array}{c} \{yxe_i^n \notin \operatorname{Im} d_n \text{ for some } 1 \leq i \leq n-1\} \\ \Updownarrow \\ \left\{-q + (-1)^n q^i \alpha = 0 \text{ and } q^{n-i} + (-1)^{n+1} q\beta = 0\right\}, \end{array}$$

and when the characteristic of k is not 2 this happens precisely when we have the following:

$$\text{(C2)} \qquad s \leq 0, \quad t \geq 0, \quad \alpha = (-1)^{t-s} q^s, \quad \beta = (-1)^{t-s} q^t.$$

In the characteristic 2 case we may relax this condition; in this case $(*)$ and $(**)$ occur precisely when we have the following:

$$\text{(C3)} \qquad s \leq 0, \quad t \geq 0.$$

However, both when the characteristic of k is not 2 and (C2) holds, and in the characteristic 2 case when (C3) holds, we see that $(*)$ occurs for $n = t - s + 1$, whereas $(**)$ occurs for $n = t - s + 2$.

Therefore, when the characteristic of k is not 2 the dimension of $\operatorname{Im} d_n$ is given by

$$\dim_k \operatorname{Im} d_n = \begin{cases} 2n & \text{when (C2) holds and } n = t - s + 1 \\ 2n & \text{when (C2) holds and } n = t - s + 2 \\ 2n+1 & \text{otherwise,} \end{cases}$$

implying the dimension of $\operatorname{Ker} d_{n+1}$ is given by

$$\dim_k \operatorname{Ker} d_{n+1} = \begin{cases} 2n+2 & \text{when (C2) holds and } n = t - s \\ 2n+2 & \text{when (C2) holds and } n = t - s + 1 \\ 2n+1 & \text{otherwise.} \end{cases}$$

Consequently , the dimension of $\operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi)$ for $n > 0$ is given by

$$\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = \begin{cases} 1 & \text{when (C2) holds and } n = t - s \\ 2 & \text{when (C2) holds and } n = t - s + 1 \\ 1 & \text{when (C2) holds and } n = t - s + 2 \\ 0 & \text{otherwise.} \end{cases}$$

When the characteristic of k is 2, we obtain the exact same formulas, but with (C2) replaced by (C3).

Remark. (i) In the case considered above, we see that the cohomology is zero except possibly in three degrees, depending on what conditions s, t, α and β satisfy. As a consequence, we construct a counterexample to the following conjecture by Auslander (see [1, page 815]): if M is a finitely generated module over an Artin algebra Γ, then there exists a number n_M such that for any finitely generated module N we have

$$\operatorname{Ext}^i_\Gamma(M, N) = 0 \text{ for } i \gg 0 \Rightarrow \operatorname{Ext}^i_\Gamma(M, N) = 0 \text{ for } i \geq n_M.$$

The first counterexample to this conjecture appeared in [6], where the algebra considered was a finite dimensional commutative Noetherian local Gorenstein algebra. A counterexample over our algebra $\Lambda = k\langle x, y\rangle/(x^2, xy + qyx, y^2)$ was given in [8].

As for a counterexample using Hochschild cohomology, define for each natural number t an automorphism $\psi\colon \Lambda \to \Lambda$ by $x \mapsto q^{-t}x$ and $y \mapsto q^t y$, and denote the bimodule ${}_1\Lambda_\psi$ by M_t. Then condition (C2)/(C3) is satisfied (with $s = -t, \alpha = q^{-t}$ and $\beta = q^t$), and so

$$\operatorname{Ext}^n_{\Lambda^e}(\Lambda, M_t) = \begin{cases} \neq 0 & \text{for } n = 2t + 2 \\ 0 & \text{for } n > 2t + 2. \end{cases}$$

(ii) Even though the above conjecture of Auslander fails to hold in general, Auslander himself proved (unpublished, see [1, page 815]) that if the conjecture holds for the enveloping algebra Γ^{e} of a finite dimensional algebra Γ over a field, then the finitistic dimension

$$\sup\{\operatorname{pd}_\Gamma X \mid X \text{ finitely generated } \Gamma\text{-module with } \operatorname{pd}_\Gamma X < \infty\}$$

of Γ is finite. In view of the above remark, we see that the converse to this result does not hold; our algebra Λ, being selfinjective, trivially has finite finitistic dimension, whereas the conjecture of Auslander does not hold for Λ^{e}.

(iii) The computation of the Hochschild cohomology $\operatorname{HH}^*(\Lambda) = \operatorname{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$ of Λ is covered by the last of the above cases. When ψ is the identity automorphism we have $s = t = 0$, and the condition (C2)/(C3) is satisfied. This gives

$$\dim_k \operatorname{HH}^n(\Lambda) = \begin{cases} 2 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 0 & \text{for } n \geq 3, \end{cases}$$

and so our algebra Λ is a counterexample to the following question raised by Happel in [5]: if the Hochschild cohomology groups of a finite dimensional algebra vanish in high degrees, does the algebra have finite global dimension? The counterexample above first appeared in [2], where it was shown that the generating function $\sum_{n=0}^{\infty} \operatorname{HH}^n(\Lambda) t^n$ of $\operatorname{HH}^n(\Lambda)$ is $2 + 2t + t^2$.

The converse to the question of Happel is always true when the algebra modulo its Jacobson radical is separable over the ground field. More specifically, if Γ is a finite dimensional algebra over a field K with Jacobson radical $\mathfrak{r}$, and the semisimple algebra $\Gamma/\mathfrak{r}$ is separable over K, then by [4, §3] the implication

$$\operatorname{gl.dim} \Gamma < \infty \Rightarrow \operatorname{pd}_{\Gamma^{\mathrm{e}}} \Gamma < \infty$$

holds. In particular the Hochschild cohomology groups $\operatorname{HH}^n(\Gamma) = \operatorname{Ext}^n_{\Gamma^{\mathrm{e}}}(\Gamma, \Gamma)$ and the homology groups $\operatorname{HH}_n(\Gamma) = \operatorname{Tor}^{\Gamma^{\mathrm{e}}}_n(\Gamma, \Gamma)$ of Γ vanish for $n \gg 0$ when the global dimension of Γ is finite. It is not known whether the vanishing of the Hochschild homology groups in high degrees for a finite dimensional algebra implies the global dimension of the algebra is finite.

(iv) Because of the equality $\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = \dim_k \operatorname{HH}_n(\Lambda, {}_1\Lambda_{(\nu\psi)^{-1}})$ which follows from Section 1, the somewhat strange behavior in Hochschild cohomology revealed in the last case considered above can also be transferred to Hochschild homology. When the automorphism ψ is given by $\psi(x) = \pm q^s x$ and $\psi(y) = \pm q^t y$, then the automorphism $\theta \stackrel{\text{def}}{=} (\nu\psi)^{-1}$, where ν is the Nakayama automorphism, is given by

$$x \mapsto \mp q^{1-s} x, \quad y \mapsto \mp q^{-(t+1)} y.$$

Thus for such an automorphism θ, when $s \in \mathbb{Z} \setminus \{1\}$ and $t \in \mathbb{Z} \setminus \{-1\}$ we get

$$\dim_k \operatorname{HH}_n(\Lambda, {}_1\Lambda_\theta) = \begin{cases} 1 & \text{when (C2)/(C3) holds and } n = t - s \\ 2 & \text{when (C2)/(C3) holds and } n = t - s + 1 \\ 1 & \text{when (C2)/(C3) holds and } n = t - s + 2 \\ 0 & \text{otherwise.} \end{cases}$$

when $n > 0$. In these formulas condition (C2) applies when the characteristic of k is not 2, and condition (C3) applies in the characteristic 2 case.

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References

[1] Auslander, M. (1999). Selected works, part 1. Reiten, Smalø, Solberg (editors). American Mathematical Society.

[2] Buchweitz, R.-O., Green, E., Madsen, D., Solberg, Ø. (2005). Finite Hochschild cohomology without finite global dimension. *Math. Res. Lett.* 12(5-6):805-816.

[3] Cartan, H., Eilenberg, S. (1956). *Homological algebra*. Princeton University Press.

[4] Eilenberg, S., Ikeda, M., Nakayama, T. (1955). On the dimension of modules and algebras, I. *Nagoya Math. J.* 8:49-57.

[5] Happel, D. (1989). Hochschild cohomology of finite-dimensional algebras. In: *Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988).* Lecture Notes in Mathematics 1404:108-126.

[6] Jorgensen, D., Sega, L. (2004). Nonvanishing cohomology and classes of gorenstein rings. *Adv. Math.* 188(2):470-490.

[7] Schulz, R. (1986). Boundedness and periodicity of modules over QF rings. *J. Algebra* 101:450-469.

[8] Smalø, S. (2006). Local limitations of the Ext functor do not exist. *Bull. London Math. Soc.* 38(1):97-98.

Petter Andreas Bergh, Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway

E-mail address: bergh@math.ntnu.no